Fermions

- generators
- independent variables

- Grassmann algebra $\mathcal{G}_D$ generated by the Grassmann variables $\eta_{A=1,\dots,D}$

  $\equiv$ set of all objects obtained by taking generic polynomials of η_A with complex coefficients, and with the rule

$$\eta_A \eta_B = -\eta_B \eta_A \qquad (*)$$

Notice that $(*)$ implies, for $A=B$, $\eta_A^2 = -\eta_A^2 \Rightarrow \eta_A^2 = 0$.

$\mathcal{G}_1 = \{ a + a_1 \eta \mid a, a_1 \in \mathbb{C} \}$ only degree 1 polynomials because $\eta^2 = 0$

$\mathcal{G}_2 = \{ a + a_1 \eta_1 + a_2 \eta_2 + \underbrace{a_{12} \eta_1 \eta_2} \mid a, a_1, a_2, a_{12} \in \mathbb{C} \}$ notice $\eta_2 \eta_1 = -\eta_1 \eta_2$

$\eta_1^2 \eta_2 = 0$

$\eta_1 \eta_2 \eta_1 = -\eta_1^2 \eta_2 = 0$

$a_{21} \equiv -a_{12}$: $\quad \frac{1}{2} a_{12} \eta_1 \eta_2 - \frac{1}{2} a_{12} \eta_2 \eta_1 = \frac{1}{2} a_{12} \eta_1 \eta_2 + \frac{1}{2} a_{21} \eta_2 \eta_1 = \frac{1}{2} \sum_{AB} a_{AB} \eta_A \eta_B$

$$\mathcal{G}_D = \left\{ a + \sum_{n=1}^{D} \sum_{1 \le A_1 < A_2 < \dots < A_n \le D} a_{A_1 \cdots A_n} \eta_{A_1} \cdots \eta_{A_n} \;\middle|\; a, a_A, a_{A_1 A_2} \dots \in \mathbb{C} \right\}$$

$$= \left\{ a + \sum_{n=1}^{D} \frac{1}{n!} \sum_{A_1 \cdots A_n = 1}^{D} a_{A_1 \cdots A_n} \eta_{A_1} \cdots \eta_{A_n} \;\middle|\; \begin{array}{l} a, a_A, a_{A_1 A_2}, \dots \in \mathbb{C} \\ \text{and } a_{A_1 \cdots A_n} \text{ fully antisymmetric} \end{array} \right\}$$

- Fermions / bosons

Elements of G_D that are even / odd under $\eta_A \to -\eta_A$ are called bosons/bosonic / fermions/fermionic

Every element of G_D is uniquely written as a sum of a bosonic and a fermionc element.

$$a + \sum_{n=1}^{D} \frac{1}{n!} \sum_{A_1 \cdots A_n = 1}^{D} a_{A_1 \cdots A_n} \eta_{A_1} \cdots \eta_{A_n} = \left. a + \sum_{\substack{n=2 \\ \text{even}}}^{D} \frac{1}{n!} \sum_{A_1 \cdots A_n = 1}^{D} a_{A_1 \cdots A_n} \eta_{A_1} \cdots \eta_{A_n} \right\} \text{boson}$$

$$\left. + \sum_{\substack{n=1 \\ \text{odd}}}^{D} \frac{1}{n!} \sum_{A_1 \cdots A_n = 1}^{D} a_{A_1 \cdots A_n} \eta_{A_1} \cdots \eta_{A_n} \right\} \text{fermion}$$

Notice: if b, b_1, b_2 bosons, f, f_1, f_2 fermions:

$$b_1 b_2 = b_2 b_1 \qquad bf = fb$$
$$f_1 f_2 = - f_2 f_1$$

- Functions of Grassmann variables

Notice that every element of G_D can be written as $a+\vartheta$ where $a\in\mathbb{C}$ and ϑ contains at least one power of η. Then $\vartheta^{D+1}=0$ because every term in ϑ^{D+1} contains at least $D+1$ factors η_A (which means that at least one of them is repeated).

Let $f:\mathbb{C}\to\mathbb{C}$ be an infinitely differentiable function. $f(a+\vartheta)$ is defined by means of its Taylor expansion

$$f(a+\vartheta)\equiv\sum_{n=0}^{\infty}\frac{\vartheta^n}{n!}f^{(n)}(a)=\sum_{n=0}^{D}\frac{\vartheta^n}{n!}f^{(n)}(a)$$

E.g.

$$e^{\eta}=1+\eta$$

$$e^{\eta_1+\eta_2}=1+(\eta_1+\eta_2)+\frac{1}{2}(\eta_1+\eta_2)^2=1+\eta_1+\eta_2 \qquad \leftarrow \text{check this!}$$

$$e^{\eta_A M_{AB}\eta_B}=\sum_{n=0}^{D}\frac{1}{n!}(\eta_A M_{AB}\eta_B)^D$$

- Derivative with respect to Grassmann variables

Rules:

- $\frac{\partial}{\partial \eta_A} \frac{\partial}{\partial \eta_B} = - \frac{\partial}{\partial \eta_B} \frac{\partial}{\partial \eta_A}$

- $a \in \mathbb{C} \quad \frac{\partial}{\partial \eta_A} a = 0$

- $\frac{\partial}{\partial \eta_A} \eta_B = \delta_{AB}$

- $\frac{\partial}{\partial \eta_A} (\theta_1 \theta_2) = \begin{cases} \left(\frac{\partial}{\partial \eta_A} \theta_1\right) \theta_2 + \theta_1 \frac{\partial}{\partial \eta_A} \theta_2 & \text{if } \theta_1 \text{ is a boson} \\ \left(\frac{\partial}{\partial \eta_A} \theta_1\right) \theta_2 - \theta_1 \frac{\partial}{\partial \eta_A} \theta_2 & \text{if } \theta_1 \text{ is a fermion} \end{cases}$

E.g. $\quad \frac{\partial}{\partial \eta_2} (a + a_1 \eta_1 + a_2 \eta_2 + a_{12} \eta_1 \eta_2) = a_2 + a_{12} \frac{\partial}{\partial \eta_2} (\eta_1 \eta_2) = a_2 - a_{12} \eta_1$

- Integrals with respect to Grassmann variables

Rules:

- $d\eta_A \, d\eta_B = - d\eta_B \, d\eta_A$
- $d\eta_A \, \eta_B = - \eta_B \, d\eta_A$
- $\int d\eta_A \, \eta_A = 1 \quad , \quad \int d\eta_A \, 1 = 0$

E.g.:

$$\int d\eta_1 d\eta_2 \left(a + a_1 \eta_1 + a_2 \eta_2 + a_{12} \eta_1 \eta_2 \right) =$$

$$= a \underbrace{\int d\eta_1}_{=0} \underbrace{\int d\eta_2}_{=0} - a_1 \underbrace{\int d\eta_1 \, \eta_1}_{=1} \underbrace{\int d\eta_2}_{=0} + a_2 \underbrace{\int d\eta_1}_{=0} \underbrace{\int d\eta_2 \, \eta_2}_{=1} - a_{12} \underbrace{\int d\eta_1 \, \eta_1}_{=1} \underbrace{\int d\eta_2 \, \eta_2}_{=1} = - a_{12}$$

Integral with respect to all Grassmann variables $\eta_1 \cdots \eta_D$ - Generic element of G_D:

$$\vartheta = a + \sum_{n=1}^{D} \frac{1}{n!} \sum_{A_1 \cdots A_n} a_{A_1 \cdots A_n} \eta_{A_1} \cdots \eta_{A_n} = a + \sum_{n=1}^{D} \sum_{1 \le A_1 < A_2 < \ldots < A_n \le D} a_{A_1 \cdots A_n} \eta_{A_1} \cdots \eta_{A_D}$$

$$\int d\eta_D \cdots d\eta_1 \, \vartheta = \underbrace{\sum_{1 \le A_1 < A_2 < \ldots < A_D = D}}_{\text{only 1 term: } A_1 = 1,\ A_2 = 2,\ A_3 = 3, \ldots} a_{A_1 \cdots A_n} \int d\eta_D \cdots d\eta_1 \, \eta_{A_1} \cdots \eta_{A_D} = a_{12 \cdots D} \int d\eta_D \cdots d\eta_1 \, \eta_1 \cdots \eta_D = a_{12 \cdots D}$$

- Linear change of variables $\eta'_A = M_{AB}\eta_B$

$\eta_1 \cdots \eta_D$ set of generators for $\mathcal{G}_D$; $M = D \times D$ invertible complex matrix

$\eta'_1, \dots, \eta'_D$ is also a set of generators for $\mathcal{G}_D$, i.e. every element ϑ of $\mathcal{G}_D$ can be written uniquely as

$$\vartheta = a + \sum_{n=1}^{D} \frac{1}{n!} \sum_{A_1 \cdots A_n} a_{A_1 \cdots A_n} \eta_{A_1} \cdots \eta_{A_n} = a' + \sum_{n=1}^{D} \frac{1}{n!} \sum_{A_1 \cdots A_n} a'_{A_1 \cdots A_n} \eta'_{A_1} \cdots \eta'_{A_n}$$

(a & a' fully antisymmetric)

In fact: $a' = a$ $\qquad a_{B_1 \cdots B_n} = \sum_{A} a'_{A_1 \cdots A_n} M_{A_1 B_1} \cdots M_{A_n B_n}$

$$\int d\eta_D \cdots d\eta_1 \, \vartheta = a_{12 \cdots D} = \sum_{A} a'_{A_1 \cdots A_D} M_{A_1 1} M_{A_2 2} \cdots M_{A_D D}$$

previous slide

$$= a'_{12 \cdots D} \sum_{A} \varepsilon_{A_1 \cdots A_D} M_{A_1 1} \cdots M_{A_D D}$$

$$= a'_{12 \cdots D} \det M = \det M \int d\eta'_D \cdots d\eta'_1 \, \vartheta$$

$a_{A_1 \cdots A_D} = a_{12 \cdots D} \, \varepsilon_{A_1 \cdots A_D}$

Since $a_{A_1 \cdots A_D}$ is fully antisymmetric, then it is $\neq 0$ only if all indices take different values - D indices & D value $\Rightarrow$ each $1, 2, \dots, D$ value needs to appear once

Written as $\qquad d\eta_D \cdots d\eta_1 = \det M \; d\eta'_D \cdots d\eta'_1$

- Integration-by-parts formula $\int d\eta_B \frac{\partial}{\partial \eta_B} \vartheta = 0$ (no sum over B)

$$\vartheta = a + \sum_{n=1}^{D} \frac{1}{n!} \sum_{A_1 \cdots A_n} a_{A_1 \cdots A_n} \eta_{A_1} \cdots \eta_{A_n} = \begin{pmatrix} \text{terms containing} \\ \text{no } \eta_B \end{pmatrix} + \begin{pmatrix} \text{terms containing} \\ 1 \text{ power of } \eta \end{pmatrix}$$

$$= \chi_0 + \eta_B \chi_1$$

where χ_0 and χ_1 contain <u>no</u> η_B

$$\frac{\partial}{\partial \eta_B} \vartheta = \chi_1$$

$$\int d\eta_B \frac{\partial}{\partial \eta_B} \vartheta = \int d\eta_B \, \chi_1 = 0$$

$\hookrightarrow$ because χ_1 contains no η_B

∎

- Quark fields

$\psi = \begin{pmatrix} u \\ d \\ s \\ c \end{pmatrix}$ $\qquad \bar\psi = \begin{pmatrix} \bar u \\ \bar d \\ \bar s \\ \bar c \end{pmatrix}$

in path-integral formalism there is no relation between ψ & $\bar\psi$ (independent Grassmann variables)

$\psi_{f\alpha i}(x)$ $\qquad x \in \Lambda$ $\qquad \bar\psi_{fi\alpha}(x)$

color index $i = 1, 2, \dots, N_c$ (the gauge group acts on this; for QCD $N_c = 3$)

spin index $\alpha = 1, 2, 3, 4$ (gamma matrices act on this)

flavour index $f = 1, \dots, N_f$ (it distinguishes u, d, s, c)

We have $\qquad 2(\psi, \bar\psi) \times N_f \times 4(\text{spin}) \times N_c \times N^4(\text{number of points}) = 8 N_f N_c N^4$

independent Grassmann variables

- Fermion action and path integral

Continuum Euclidean action $S_F = \sum_f \int d^4x\, \bar\psi_f (\not{D} + m_f)\psi_f$ $\not{D} = \sum_\mu D_\mu \gamma_\mu$

γ_μ are the Euclidean gamma matrices, i.e $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$

E.g. (chiral basis) $\gamma_0 = \begin{pmatrix} 0 & -I_2 \\ -I_2 & 0 \end{pmatrix}$ $\gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix}$ $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$

General structure of the discretized action

$$S_F = \sum_f \sum_x a^4\, \bar\psi_f(x) \left[\mathcal{D}^f \psi_f\right](x)$$

For every f, $\mathcal{D}^f$ is a matrix with spin, color, lattice indices

$$\left[\mathcal{D}^f \psi_f\right]_{\alpha i}(x) = \sum_{j\beta y} \mathcal{D}^f_{\underbrace{\alpha i x}; \underbrace{\beta j y}} \psi_{\beta j}(y)$$

interpret this as a single index

$\mathcal{D}^f$ is a discretization of the operator $\not{D} + m_f$ and is called discrete Dirac operator [we will specify it later]

$\mathcal{D}^f$ is a $(4N_cN^4) \times (4N_cN^4)$ matrix

Path integral measure over fermions:

$$[d\bar{\psi} d\psi] \equiv \prod_{x\in\Lambda} \prod_{\alpha=1}^{4} \prod_{i=1}^{N_c} \prod_{f=1}^{N_f} d\bar{\psi}_{f\alpha i}(x)\, d\psi_{f\alpha i}(x)$$

Fermion partition function:

$$Z = \int [d\bar{\psi} d\psi]\, e^{-\sum_f \sum_x a^4 \bar{\psi}_f(x) \mathcal{D}^f \psi(x)} = \prod_f \int [d\bar{\psi}_f d\psi_f]\, e^{-\sum_x a^4 \bar{\psi}_f(x) \mathcal{D}^f \psi_f(x)}$$

change of variable $\psi' = a^4 \mathcal{D}^f \psi$ $\qquad [d\psi_f d\bar{\psi}_f] = \det(a^4 \mathcal{D}^f)\, [d\psi'_f d\bar{\psi}_f]$

$$= \prod_f \left\{ \det(a^4 \mathcal{D}^f) \underbrace{\int [d\bar{\psi}_f\, d\psi'_f]\, e^{-\sum_x \bar{\psi}_f(x) \psi'_f(x)}}_{\prod_{x i \alpha} \int d\bar{\psi}_{f\alpha i}\, d\psi'_{f\alpha i}\, e^{-\bar{\psi}_{f\alpha i} \psi'_{f\alpha i}}} \right\} = \prod_f \det(a^4 \mathcal{D}^f)$$

$$\prod_{x i \alpha} \int d\bar{\psi}_{f\alpha i}\, d\psi'_{f\alpha i}\, e^{-\bar{\psi}_{f\alpha i} \psi'_{f\alpha i}} = \left(\int d\bar{\eta} d\eta\, e^{-\bar{\eta}\eta} \right)^{4N_c N^3}$$

$$= \left[\int d\bar{\eta} d\eta\, (1 - \bar{\eta}\eta) \right]^{4N_c N^3} = 1$$

Fermion 2-pt function $\quad \langle \psi_{f\alpha i}(x)\, \bar\psi_{f'\beta j}(y) \rangle = \delta_{ff'}\, (a^4 D^f)^{-1}_{\alpha i x;\, \beta j y}$

also written as $\quad \langle \psi_f(x)\, \bar\psi_{f'}(y) \rangle = \delta_{ff'}\, (a^4 D^f)^{-1}(x,y)$

Let's prove this. We start from the identity

$$0 = \int [d\bar\psi\, d\psi]\, \frac{\partial}{\partial \bar\psi_{f\alpha i}(x)} \left\{ e^{-\sum_g \sum_z a^4 \bar\psi_g(z) D^g \psi_g(z)}\, \bar\psi_{f'\beta j}(y) \right\}$$

$$= \int [d\bar\psi\, d\psi]\, e^{-\sum_f \sum_z a^4 \bar\psi(z) D^f \psi(z)} \left\{ -(a^4 D^f \psi_f)_{\alpha i}(x)\, \bar\psi_{f'\beta j}(y) + \delta_{ff'}\, \delta_{\alpha\beta}\, \delta_{ij}\, \delta_{xy} \right\}$$

Divide by Z: $\quad 0 = -\langle (a^4 D^f \psi_f)_{\alpha i}(x)\, \bar\psi_{f'\beta j}(y) \rangle + \delta_{ff'}\, \delta_{\alpha\beta}\, \delta_{ij}\, \delta_{xy}$

i.e. $\quad (a^4 D^f) \langle \psi_f\, \bar\psi_{f'} \rangle = \delta_{ff'}\, I_{\text{spin, color, coord}}$

$\quad \langle \psi_f\, \bar\psi_{f'} \rangle = \delta_{ff'}\, (a^4 D^f)^{-1}$ (matrix identity)

$\blacksquare$

Free fermions and doubler problem

$$S_F = \sum_x a^4 \bar{\psi}(x) D \psi(x)$$

(One flavour Free fermions)

D is a discretization of $\not{\partial} + m$

Various options. The simplest ones:

(a) $S_F = \sum_x a^4 \bar{\psi} \left\{ \sum_\mu \gamma_\mu \partial^f_\mu + m \right\} \psi$ → it breaks C & P & Euclidean time reversal

(b) $S_F = \sum_x a^4 \bar{\psi} \left\{ \sum_\mu \gamma_\mu \partial^b_\mu + m \right\} \psi$ → it breaks C & P & Euclidean time reversal

(c) $S_F = \sum_x a^4 \bar{\psi} \left\{ \sum_\mu \gamma_\mu \partial^s_\mu + m \right\} \psi$ → doublers

$$\partial^s_\mu = \frac{\partial^f_\mu + \partial^b_\mu}{2}$$

$$\partial^s_\mu \varphi(x) = \frac{\varphi(x+a e_\mu) - \varphi(x - a e_\mu)}{2a}$$

symmetric derivative

(a) $S_F = \sum_x a^4 \bar{\psi} \{ \sum_\mu \gamma_\mu \partial^f_\mu + m \} \psi$ Let's show that this breaks parity.

Parity acts like $\psi(x_0, \underline{x}) \to \gamma_0 \psi(x_0, -\underline{x})$ $\bar{\psi}(x_0, \underline{x}) \to \bar{\psi}(x_0, -\underline{x})\gamma_0$

In the continuum $\partial_0 \psi(x_0, \underline{x}) \to \gamma_0 \partial_0 \psi(x_0, -\underline{x})$

$\underline{\partial} \psi(x_0, \underline{x}) \to -\gamma_0 \underline{\partial} \psi(x_0, -\underline{x})$

$\gamma_\mu \partial_\mu \psi(x_0, \underline{x}) \to \gamma_0^2 \partial_0 \psi(x_0, -\underline{x}) - \underline{\gamma} \gamma_0 \underline{\partial} \psi(x_0, -\underline{x}) = \gamma_0 \gamma_\mu \partial_\mu \psi(x_0, -\underline{x})$

$\bar{\psi}(x_0, \underline{x}) (i\not{\partial} + m) \psi(x_0, \underline{x}) = \bar{\psi}(x_0, -\underline{x}) \gamma_0^2 (i\not{\partial} + m) \psi(x_0, -\underline{x})$

$S_F \to S_F$ (Change $\underline{x} \to -\underline{x}$ in the integral, and $\gamma_0^2 = 1$)

On the lattice $\partial_0^f \psi(x_0, \underline{x}) \to \gamma_0 \partial_0^f \psi(x_0, -\underline{x})$ same as continuum

but $\underline{\partial}^f \psi(x_0, \underline{x}) \to \gamma_0 \partial_0^b \psi(x_0, -\underline{x})$ (check this!)

$S_F \to \sum_x a^4 \bar{\psi} \{ \gamma_0 \partial_0^f + \sum_k \gamma_k \partial_k^b + m \} \psi \neq S_F$

Case (b) works similarly!

(c) $S_F = \sum_x a^4 \bar{\psi} \left\{ \sum_\mu \gamma_\mu \partial^s_\mu + m \right\} \psi$

Let's calculate the 1-particle energy levels. This is done by looking at the poles in p_0 of the 2pt-function calculated with $T=\infty$. (We have done the same for the scalar field)

$$\langle \psi(0)\bar{\psi}(x)\rangle = (a^4 D)^{-1}(0,x)$$

inverse is calculated by (partial) diagonalization

$$v_p(x) = a^4 e^{ipx} \qquad p_0 \in \frac{2\pi}{T}\mathbb{Z}, \quad p_k \in \frac{2\pi}{L}\mathbb{Z}, \quad -\frac{\pi}{a} < p_\mu \leq \frac{\pi}{a}$$

$$\partial^s_\mu v_p(x) = a^4 \frac{e^{ip(x+ae_\mu)} - e^{ip(x-ae_\mu)}}{2a} = \frac{e^{iap_\mu} - e^{-iap_\mu}}{2a} v_p(x) = \frac{i}{a}\sin(ap_\mu)\, v_p(x)$$

$$D v_p = \left(\sum_\mu \gamma_\mu \partial^s_\mu + m\right) v_p = \underbrace{\left\{ \sum_\mu \frac{i}{a}\gamma_\mu \sin(ap_\mu) + m \right\}}_{\left\{\sum_\mu i\mathring{p}_\mu \gamma_\mu + m\right\} \text{ 4x4 matrix}} v_p$$

[D is block diagonal in p space]

$$D^{-1} v_p = \left\{ \sum_\mu i\mathring{p}_\mu \gamma_\mu + m \right\}^{-1} v_p$$

the inverse of this 4x4 matrix can be explicitly calculated

$$\not{D}^{-1} v_p = \frac{1}{\sum_\mu i\mathring{p}_\mu \gamma_\mu + m} v_p = \left(-\sum_\mu i\mathring{p}_\mu \gamma_\mu + m\right) \frac{1}{\left(\sum_\mu i\mathring{p}_\mu \gamma_\mu + m\right)\left(-\sum_\mu i\mathring{p}_\mu \gamma_\mu + m\right)} v_p$$

$A = \sum_\mu \mathring{p}_\mu \gamma_\mu$:

$$\text{den} = (iA+m)(-iA+m) = A^2 + imA - imA + m^2 = A^2 + m^2$$

$$= \sum_{\mu\nu} \mathring{p}_\mu \mathring{p}_\nu \gamma_\mu \gamma_\nu + m^2 = \sum_{\mu\nu} \mathring{p}_\mu \mathring{p}_\nu \left(\frac{1}{2}\gamma_\mu\gamma_\nu + \frac{1}{2}\gamma_\nu\gamma_\mu\right) + m^2$$

$$= \sum_{\mu\nu} \mathring{p}_\mu \mathring{p}_\nu \frac{1}{2}\{\gamma_\mu, \gamma_\nu\} + m^2 = \sum_\mu \mathring{p}_\mu^2 + m^2 \rightarrow \text{multiple of identity!}$$

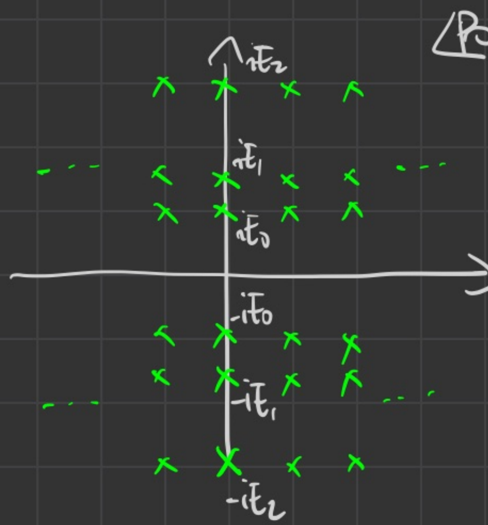
$$\not{D}^{-1} v_p = \frac{-i\mathring{\not{p}} + m}{\sum_\mu \mathring{p}_\mu^2 + m^2} v_p$$

use completeness relation $I = \frac{1}{a^4 L^3 T} \sum_p v_p v_p^\dagger$

$$(a^4 \not{D})^{-1} = \frac{1}{a^8 L^3 T} \sum_p \frac{-i\mathring{\not{p}} + m}{\sum_\mu \mathring{p}_\mu^2 + m^2} v_p v_p^\dagger$$

$$\langle \psi(0) \bar{\psi}(x) \rangle = (a^4 \not{D})^{-1}(0,x) = \frac{1}{L^3 T} \sum_p \frac{-i\mathring{\not{p}} + m}{\sum_\mu \mathring{p}_\mu^2 + m^2} e^{-ipx}$$

Discrete Fourier transform in x_0, and $T\to\infty$ limit



$$\lim_{T\to\infty} \sum_{x_0} a\, e^{ip_0x_0} \langle \eta(0)\bar\eta(x)\rangle = \frac{1}{L^3}\sum_{\underline{p}} \frac{-i\mathring{\not{p}}+m}{\sum_\mu \mathring{p}_\mu^2+m^2}\, e^{-i\underline{p}\,\underline{x}}$$

Poles:

$$0=\sum_\mu \mathring{p}_\mu^2+m^2 = \frac{1}{a^2}\sum_\mu \sin^2(ap_\mu)+m^2$$

(solve for complex p_0)

$$\sin^2(ap_0) = -a^2m^2 - \sum_k \sin^2(ap_k)^2$$

$$\sin(ap_0) = \pm i\sqrt{a^2m^2 - \sum_k \sin^2(ap_k)^2} \equiv \pm i\omega(\underline{p})$$

$$ap_0 = \pm \sin^{-1}[i\omega(\underline{p})] + \qquad , \quad n\in\mathbb{Z}$$

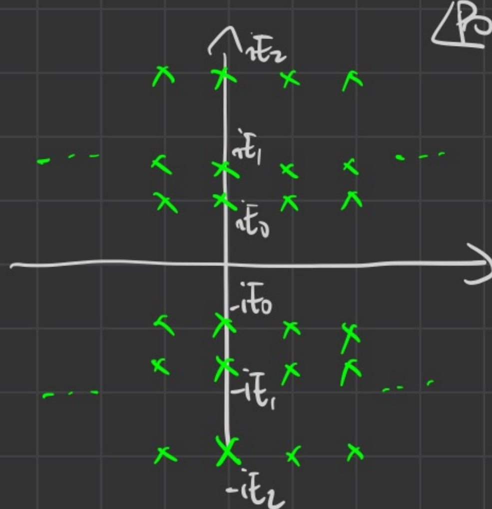
identity $\quad \sin(ix) = \frac{e^{i(ix)}-e^{-i(ix)}}{2i} = \frac{e^{-x}-e^{x}}{2i} = i\sinh(x)$

$x=\sinh^{-1}(\omega)$: $\quad \sin(i\sinh^{-1}(\omega)) = i\omega$

$\qquad i\sinh^{-1}(\omega) = \sin^{-1}(i\omega)$

$$p_0 = \pm\frac{i}{a}\sinh^{-1}\omega(\underline{p}) + \frac{2\pi n}{a} \qquad \text{or} \qquad p_0 =$$

Discrete Fourier transform in x_0, and $T\to\infty$ limit



$$\lim_{T\to\infty} \sum_{x_0} a\, e^{ip_0x_0} \langle \eta(0)\bar{\eta}(x)\rangle = \frac{1}{L^3}\sum_{\underline{p}} \frac{-i\mathring{\not{p}}+m}{\sum_\mu \mathring{p}_\mu^2+m^2}\, e^{-i\underline{p}\,\underline{x}}$$

Poles: $0 = \sum_\mu \mathring{p}_\mu^2 + m^2 = \frac{1}{a^2}\sum_\mu \sin^2(ap_\mu) + m^2$ (solve for complex p_0)

$$\sin^2(ap_0) = -a^2m^2 - \sum_k \sin^2(ap_k)^2$$

identity: $\sinh(ix) = \frac{e^{ix}-e^{-ix}}{2} = i\sin(x)$

periodic $ap_0 \to ap_0 + \pi n$ $\longleftarrow$ $\sinh^2(iap_0) = -\sin^2(ap_0) = am^2 + \sum_k \sin^2(ap_k)$ $\quad n\in\mathbb{Z}$

$$iap_0 = \pm \operatorname{arcsinh}\sqrt{a^2m^2 + \sum_k \sin^2(ap_k)} - i\pi n$$

periodicity is $\frac{\pi}{a}$ it should be $\frac{2\pi}{a}$! $\longleftarrow$ $p_0 = \mp \frac{i}{a}\operatorname{arcsinh}\sqrt{a^2m^2 + \sum_k \sin^2(ap_k)} + \frac{\pi}{a}n$

$$E(\underline{p}) = \frac{1}{a}\operatorname{arcsinh}\sqrt{a^2m^2 + \sum_k \sin^2(ap_k)} \longrightarrow$$

TOO MANY STATES!

$\underline{p} = (0,0,0)$ $\quad \underline{p} = (\frac{\pi}{a},0,0)$

$\underline{p} = (0,\frac{\pi}{a},0)$ $\quad \underline{p} = (0,0,\frac{\pi}{a})$

$\underline{p} = (\frac{\pi}{a},\frac{\pi}{a},0)$ $\quad \underline{p} = (\frac{\pi}{a},0,\frac{\pi}{a})$

$\underline{p} = (0,\frac{\pi}{a},\frac{\pi}{a})$ $\quad \underline{p} = (\frac{\pi}{a},\frac{\pi}{a},\frac{\pi}{a})$

all give the same energy

$E(\underline{p}) = \frac{1}{a}\operatorname{arcsinh}(am) \Rightarrow$ 8 doublers!

Summary: doublers with naive fermions $D = i\mathring{\not{p}} + m$

The 2pt function $\lim_{T\to\infty} \sum_{x_0} a\, e^{ip_0 x} \langle \psi(0)\bar{\psi}(x)\rangle$ has poles at:

$$p_0 = \pm i E(\mathbf{p}) + \frac{\pi}{a} n \quad \text{with } n \in \mathbb{Z} \qquad -\frac{\pi}{a} < p_k \le \frac{\pi}{a} \qquad \left[\begin{array}{ll} \text{finite } L: & p_k \in \frac{2\pi}{L}\mathbb{Z} \\ L=\infty: & p_k \in \mathbb{R} \end{array}\right]$$

$$E(\mathbf{p}) = \frac{1}{a}\arcsin\sqrt{a^2m^2 + a^2\sum_k \mathring{p}_k^2} \qquad \text{with } \mathring{p}_\mu = \frac{1}{a}\sin(ap_\mu)$$

= energy of a single particle state with momentum $\mathbf{p}$

Obs 1: p_0 has the wrong periodicity. It is $\frac{\pi}{a}$, it should be $\frac{2\pi}{a}$.

Given p_k, define $p_k' = \begin{cases} \frac{\pi}{a} - p_k & \text{if } 0 \le p_k \le \frac{\pi}{a} \\ -\frac{\pi}{a} - p_k & \text{if } -\frac{\pi}{a} < p_k < 0 \end{cases}$. This satisfies

- $-\frac{\pi}{a} < p_k' \le \frac{\pi}{a}$
- $\lim_{\substack{a\to 0 \\ p_k \text{ const.}}} |p_k'| = \infty$

Notice that $\mathring{p}_k' = \frac{1}{a}\sin(\pm\pi - ap_k) = \frac{1}{a}\sin(ap_k) = \mathring{p}_k$

$E(\mathbf{p})$ does not change under $p_k \to p_k'$ (separately for every component)

Obs 2: Given a state with momentum $\mathbf{p} = (p_1, p_2, p_3)$, there are 7 other degenerate states (doublers) with momenta

(p_1', p_2, p_3) (p_1, p_2', p_3) (p_1, p_2, p_3') (p_1', p_2', p_3) (p_1', p_2, p_3') (p_1, p_2', p_3') (p_1', p_2', p_3')

Notice that the momentum of the doublers $\to \infty$ for $a \to 0$, while their energy stays finite!

Wilson fermions

Wilson - Dirac operator

$$D = \sum_\mu \gamma_\mu \partial^s_\mu + m - \frac{ar}{2} \sum_\mu \partial^b_\mu \partial^f_\mu$$

r = Wilson parameter [dimensionless]
common choice $r=1$

Statement: the Wilson - Dirac operator solves the doublers problem. Let's see how...

$$D v_p = [\dots] = \Big\{ \underbrace{i\mathring{\not p} + m}_{\text{naive Dirac operator}} + \frac{ar}{2} \underbrace{\hat{p}^2}_{\text{(-1) discrete Laplacian (see scalar theory)}} \Big\} v_p$$

$$\mathring{p}_\mu = \frac{1}{a}\sin(a p_\mu) \qquad \mathring{p}^2 = \sum_\mu \mathring{p}_\mu^2 \qquad \mathring{\not p} = \sum_\mu \mathring{p}_\mu \gamma_\mu$$

$$\hat{p}_\mu = \frac{2}{a}\sin\left(\frac{a p_\mu}{2}\right) \qquad \hat{p}^2 = \sum_\mu \hat{p}_\mu^2$$

$$D^{-1} v_p = \underbrace{[\dots]}_{\text{as for naive fermions}} = \frac{-i\mathring{\not p} + m + \frac{ar}{2}\hat{p}^2}{\mathring{p}^2 + \left(m + \frac{ar}{2}\hat{p}^2\right)^2}$$

from naive to Wilson

$$m_0 \longrightarrow m_0 + \frac{ar}{2}\hat{p}^2$$

$$\langle \psi(0)\bar{\psi}(x)\rangle = (a^4 D)^{-1}(0,x) = \frac{1}{L^3 T}\sum_p \frac{-i\mathring{\not p} + m + \frac{ar}{2}\hat{p}^2}{\mathring{p}^2 + \left(m + \frac{ar}{2}\hat{p}^2\right)^2} e^{-ipx}$$

Discrete Fourier transform in x_0, and $T\to\infty$ limit

$$\lim_{T\to\infty} \sum_{x_0} a\, e^{ip_0 x_0} \langle \eta(0)\bar\eta(x)\rangle = \frac{1}{L^3}\sum_{\mathbf{p}} \frac{-i\mathring{\not p} + m + \frac{ar}{2}\hat p^2}{\mathring p^2 + (m+\frac{ar}{2}\hat p^2)^2}\, e^{-i\mathbf{p}\cdot\mathbf{x}}$$

$\hat p_0 = \frac{2}{a}\sin\left(\frac{ap_0}{2}\right)$;

$\mathring p_0^2 = \frac{1}{a^2}\sin^2(ap_0) = \frac{4}{a^2}\sin^2\left(\frac{ap_0}{2}\right)\cos^2\left(\frac{ap_0}{2}\right)$

$= \hat p_0^2\left[1-\sin^2\left(\frac{ap_0}{2}\right)\right]^2 = \hat p_0^2\left(1-\frac{a^2\hat p_0^2}{4}\right)$

Poles: $\mathring p^2 + \left(m+\frac{ar}{2}\hat p^2\right)^2 = 0$ (solve for p_0)

$$\mathring p_0^2 + \sum_k \mathring p_k^2 + \left(m + \frac{ar}{2}\hat p_0^2 + \frac{ar}{2}\sum_k \hat p_k^2\right)^2 = 0$$

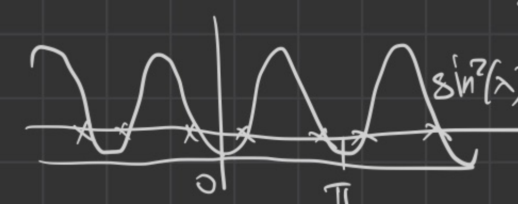
$$\hat p_0^2\left(1-\frac{a^2\hat p_0^2}{4}\right) + \sum_k \mathring p_k^2 + \frac{a^2r^2}{4}\hat p_0^4 + ar\hat p_0^2\left(m+\frac{ar}{2}\sum_k\hat p_k^2\right) + \left(m+\frac{ar}{2}\sum_k \hat p_k\right)^2 = 0$$

→ • quadratic in $\hat p_0^2$
• one can solve it in general, but $r=1$ is simpler
• let's look at this case

$r=1$

$$\hat p_0^2\left(1+am+\frac{a^2}{2}\sum_k\hat p_k^2\right) = -\left(m+\frac{a}{2}\sum_k\hat p_k^2\right)^2$$

$$\frac{4}{a^2}\sin^2\left(\frac{p_0 a}{2}\right) = \hat p_0^2 = -\frac{\left(m+\frac{a}{2}\sum_k\hat p_k^2\right)^2}{1+am+\frac{a^2}{2}\sum_k\hat p_k^2}$$



periodic under $\frac{p_0 a}{2}\to\frac{p_0 a}{2}+\pi$

$p_0\to p_0+\frac{2\pi}{a}$

(correct periodicity)

$$\sin^2\left(\frac{p_0 a}{2}\right) = -\frac{\left(am+\frac{a^2}{2}\sum_k\hat p_k^2\right)^2}{4\left(1+am+\frac{a^2}{2}\sum_k\hat p_k^2\right)}$$

$$\frac{p_0 a}{2} = \pm\arcsin\left\{ i\,\frac{am+\frac{a^2}{2}\sum_k\hat p_k^2}{2\left(1+am+\frac{a^2}{2}\sum_k\hat p_k^2\right)^{1/2}}\right\} + \pi n$$

$i\sin(x) = \sinh(ix)$

$\operatorname{arcsinh}(iz) = i\arcsin(z)$

$$p_0 = \pm i E(\mathbf{p}) + \frac{2\pi}{a} n \,, \qquad n \in \mathbb{Z} \qquad\qquad -\frac{\pi}{a} < p_\mu \leq \frac{\pi}{a}$$

$$E(\mathbf{p}) = \frac{2}{a} \operatorname{arcsinh} \left\{ \frac{am + \frac{a^2}{2} \sum_\mu \hat{p}_\mu^2}{2\left(1 + am + \frac{a^2}{2} \sum_\mu \hat{p}_\mu^2\right)^{1/2}} \right\} \qquad\qquad \hat{p}_\mu = \frac{2}{a} \sin\left(\frac{a p_\mu}{2}\right)$$

Example:

8 states

$\mathbf{p} = (0,0,0)$ $\qquad \hat{\mathbf{p}} = (0,0,0)$ $\qquad E(\mathbf{p}) = \frac{2}{a} \operatorname{arcsinh} \left\{ \frac{am}{2(1+am)^{1/2}} \right\} \xrightarrow{a \to 0} m$

$\mathbf{p} = \left(\frac{\pi}{a}, 0, 0\right)$ & 2 permutations $\qquad \hat{\mathbf{p}} = \left(\frac{2}{a}, 0, 0\right)$ $\qquad E(\mathbf{p}) = \frac{2}{a} \operatorname{arcsinh} \left\{ \frac{am + 2}{2(am + 3)^{1/2}} \right\} \xrightarrow{a \to 0} \infty$

$\mathbf{p} = \left(\frac{\pi}{a}, \frac{\pi}{a}, 0\right)$ & 2 permutations $\qquad \hat{\mathbf{p}} = \left(\frac{2}{a}, \frac{2}{a}, 0\right)$ $\qquad E(\mathbf{p}) = \frac{2}{a} \operatorname{arcsinh} \left\{ \frac{am + 4}{2(am+5)^{1/2}} \right\} \xrightarrow{a \to 0} \infty$

$\mathbf{p} = \left(\frac{\pi}{a}, \frac{\pi}{a}, \frac{\pi}{a}\right)$ $\qquad \hat{\mathbf{p}} = \left(\frac{2}{a}, \frac{2}{a}, \frac{2}{a}\right)$ $\qquad E(\mathbf{p}) = \frac{2}{a} \operatorname{arcsinh} \left\{ \frac{am + 6}{2(am+7)^{1/2}} \right\} \xrightarrow{a \to 0} \infty$

The doublers have infinite energy in the continuum limit, i.e. they decouple. Only 1 state survives for $a \to 0$!

Summary: Wilson fermions with $r=1$

$$D = i\not{D} + m - \frac{a}{2}\sum_\mu \partial_\mu^b \partial_\mu^f$$

The 2pt function $\lim_{T\to\infty} \sum_{x_0} a\, e^{ip_0 x} \langle \psi(0)\bar\psi(x)\rangle$ has poles at:

$$p_0 = \pm i E(\underline{p}) + \frac{2\pi}{a} n \qquad \text{with } n\in\mathbb{Z} \qquad -\frac{\pi}{a} < p_k \le \frac{\pi}{a}$$

$\left[\begin{array}{ll} \text{finite } L: & p_k \in \frac{2\pi}{L}\mathbb{Z} \\ L=\infty: & p_k \in \mathbb{R} \end{array}\right]$

$$E(\underline{p}) = \frac{1}{a}\operatorname{arcsin}\left\{ \frac{am + \frac{a^2}{2}\sum_k \hat{p}_k^2}{2\left(1+am+\frac{a^2}{2}\sum_k \hat{p}_k^2\right)^{1/2}} \right\} \qquad \text{with} \qquad \hat{p}_\mu = \frac{2}{a}\sin\left(\frac{ap_\mu}{2}\right)$$

= energy of a single particle state with momentum $\underline{p}$

Obs 1: p_0 has the right periodicity.

Given p_k, define $p_k' = \begin{cases} \frac{\pi}{a} - p_k & \text{if } 0\le p_k \le \pi/a \\ -\frac{\pi}{a} - p_k & \text{if } -\frac{\pi}{a} < p_k < 0 \end{cases}$. This satisfies:

- $-\frac{\pi}{a} < p_k' \le \frac{\pi}{a}$
- $\lim_{\substack{a\to 0 \\ p_k \text{ const.}}} |p_k'| = \infty$

Notice that $\hat{p}_k' = \frac{2}{a}\sin\left(\pm\frac{\pi}{2} - \frac{ap_k}{2}\right) = \pm\frac{2}{a}\cos\left(\frac{ap_k}{2}\right)$

Obs 2:
(Check calculation)

$$\lim_{\substack{a\to 0 \\ p \text{ const}}} E(\underline{p}) = \sqrt{m^2+\underline{p}^2}$$

$$\lim_{\substack{a\to 0 \\ p \text{ const}}} E(p_1', p_2, p_3) = \lim_{\substack{a\to 0 \\ p \text{ const}}} E(p_1, p_2', p_3) = [\text{all doublers}] = +\infty$$

The doublers get infinite energy (for $a\to 0$) and decouple.

Nielsen - Ninomiya theorem

A disretization of the massless free Dirac operator with the following properties does <u>not</u> exist:

(a) D is translational invariant, i.e. $D_{\alpha\beta}(x,y) = D_{\alpha\beta}(x-y,0)$

(b) $\{\gamma_5, D\} = 0$ and $D^\dagger = D$

(c) $D_{\alpha\beta}(z,0)$ decays faster than any inverse power of $|z|$ for $|z| \to \infty$

(d) D is free of doublers

Comments on assumptions:

- (b) is verifyied in the continuum: $\{\gamma_5, i\not{\partial}\} = 0$, $(i\gamma_\mu\partial_\mu)^\dagger = (-i)\gamma_\mu(-\partial_\mu) = i\gamma_\mu\partial_\mu$ and it is equivalent to the statement that the massless D is a linear combination of the γ_μ matrices, i.e.
$$D_{\alpha\beta}(z,0) = \sum_\mu (\gamma_\mu)_{\alpha\beta} F_\mu(z) \quad \text{with } F^*_\mu(z) = F_\mu(-z)$$

- (c) is a weak form of locality. Notice that for the naive Dirac operator $D_{\alpha\beta}(z,0) = 0$ if $|z| > 2a$.

Wilson for $m=0$ breaks (b) while being ultralocal

Staggered breaks (a)

GW $\left(\begin{smallmatrix}\text{overlap/} \\ \text{DW}\end{smallmatrix}\right)$ breaks (b) while being local in a weaker sense